(r,t)-RS graph: bipartite

Disjoint union of t induced matchings. of size r.

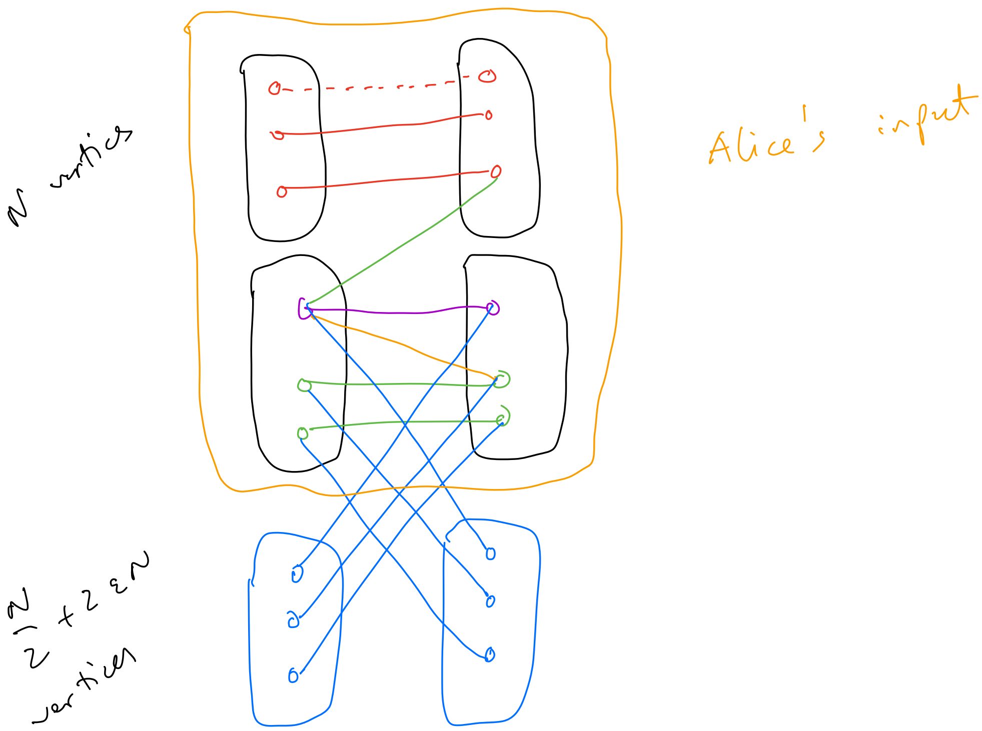
Prop: For any const $\varepsilon > 0$, $\exists$ an (r,t) RS-graph where $r = (1/4 - \varepsilon) n$

$t = n^{\Omega_\varepsilon(1/\lg\lg n)}$

Hard distribution: μ

$G^{rs} := (L^{rs}, R^{rs}, E^{rs})$ w/ N vertices

- For each M_i^{rs}, independently remove εN to form M_i
- Alice gets $M_1 \cup \dots \cup M_t$.
- Pick $j \in [n]$ uar.
- Create 2 new set of vertices L_B, R_B
- of size $N/4 + \varepsilon N$.
- Bob gets a perfect matching b/w L_B, R_B and vertices in G^{rs} that are not in M_j^{rs}.



<u>Claim</u>: The following hold.

i) The graph has a matching of size

$$\geq \left(\frac{N}{2} + \varepsilon N\right) + \left(\frac{N}{4} - \varepsilon N\right) = \frac{3N}{4}$$

ii) Any matching of size

$\frac{N}{2} + 2\varepsilon N + b$ must have at least

b edges from $M_{\bar{J}}$

<u>pf</u>:

ii) Let M be a matching of size $\frac{N}{2} + 2\varepsilon N + b$ with $< b$ edges from $M_{\bar{J}}$.

let $X =$ Bob's edges
$Y =$ M_i edges for $i \neq j$
for each edge in Y, we can replace it with an edge in X.
(induced matching notion used here).

so $|M| \leq |X| +$ #edges used in M_j.
$< \frac{N}{2} + 2\varepsilon N + b \Rightarrow$ contradiction.

Consider a deterministic protocol π

Def: $\tau(\Pi) =$ Alice's graphs mapped to msg Π

Def: an edge uv belongs to Π if $e \in G$.
$\forall G$ mapped to Π,

Def: $\tau(\Pi)_i = \{$edges in M_i^{rs} that belong to $\Pi\}$.

$$\frac{\text{Bob's output}}{\text{max-matching}} \leq \frac{N/2 + 2\varepsilon N + |\tau(\Pi)_j|}{3N/4}$$

$$\leq \frac{2}{3} + 3\varepsilon + \frac{|\tau(\Pi)_j|}{3N/4}.$$

Assumption: Bob can only output edges from the input graphs.

Assume $c = o(r, t)$

Claim: let c be the msg length.

w.p. $\frac{1}{2^c}$,

$$|\tau(\Pi)| \geq \frac{\binom{r}{\varepsilon N}^t}{2^{2c}}$$

pf: there are $\binom{r}{\varepsilon N}^t$ inputs, partitioned into 2^c msg.

since the dist π is uniform, the prob. that a msg Π is sent is $\frac{|\tau(\Pi)|}{\binom{r}{\varepsilon N}^t}$. Therefore,

$$\mathbb{P}_{\Pi}\left[|\tau(\Pi)| < \frac{\binom{r}{\varepsilon N}^t}{2^{2c}} \right] \leq \sum_{\substack{\Pi: \\ |\tau(\Pi)| < \frac{\binom{r}{\varepsilon N}^t}{2^{2c}}}} \frac{\binom{r}{\varepsilon N}^t}{2^{2c}\binom{r}{\varepsilon N}^t}$$

$$\leq 2^c \cdot \frac{1}{2^{2c}} = \frac{1}{2^c}.$$

If $|\tau(\Pi)| \geq \frac{\binom{r}{\varepsilon N}^t}{2^{2c}}$, we say Π is large.

Claim: $\forall$ large msg Π, at least $0.9t$ indices $i \in [t]$ satisfy:

$$|\tau(\Pi)_i| < \varepsilon N.$$

pf: Suppose otherwise. Then

$$|\tau(\Pi)| \le \binom{r}{\varepsilon N}^{0.9t} \cdot \binom{r-\varepsilon N}{\varepsilon N}^{0.1t}.$$

fact: $\binom{a-b}{c} \le 2^{-\Theta(b)} \cdot \binom{a}{c}$ for $c = \Theta(b)$.

$$\le \binom{r}{\varepsilon N}^{0.9t} \cdot \binom{r}{\varepsilon N}^{0.1t} \cdot 2^{-\Theta(\varepsilon N)\cdot(0.1)t}$$

(ε is const)

$$= \binom{r}{\varepsilon N}^{t} 2^{-\Omega(Nt)}$$

$$\le \frac{\binom{r}{\varepsilon N}^{t}}{2^{\Omega(Nt)}} < \frac{\binom{r}{\varepsilon N}^{t}}{2^{c}}$$ $\times$ contradiction

since we assume $c = o(rt) = o(Nt)$

<u>Claim</u>: wp ≥ 0.8, $|\tau(\Pi)_J| < \varepsilon N$.

pf: w.p $1 - \frac{1}{2^c}$, Π is large.

conditioning on large Π, the prob that Bob recovers more than εN edges from M_J^{rs} is $\le 1/10$.

hence $|\tau(\Pi)_J| \ge \varepsilon N$ wp. at most

$$\frac{1}{2^c} + \left(1 - \frac{1}{2^c}\right) \cdot \frac{1}{10} < 0.2.$$

Hence, if $c = o(r \cdot t)$ then

$$\frac{\text{Bob's output}}{\text{max matching}} < \frac{\frac{N}{2} + 2\varepsilon N + \varepsilon N}{3N/4}$$

$$= \frac{2}{3} + 5\varepsilon + \frac{\varepsilon N}{3N/4}$$

$$< \frac{2}{3} + 5 \cdot \varepsilon \cdot N$$

Reparametrizing ε, N gives.

$\boxed{\begin{array}{l}\underline{\text{Thm}}: \text{ Any 1-pass, } \left(\frac{2}{3}+\varepsilon\right) \text{ computation of the max matching must use} \\ \Omega(rt) = \Omega\left(N^{1+\Omega_\varepsilon(1/\lg\lg N)}\right) \text{ space.}\end{array}}$